

To Professor R. Cristescu on the occasion of his 70-th birthday

d-independence and d-bases in vector lattices

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1. INTRODUCTION

This article contains the results of two types. In Section 3 we give a complete characterization of band preserving projection operators on Dedekind complete vector lattices. These operators were instrumental in our work [AK2], and now we have obtained their description. This is done in Theorem 3.4. Let us mention also Theorem 3.2 that contains a description of such operators on arbitrary laterally complete vector lattices. The central role in these descriptions is played by d-bases, one of two principal tools utilized in [AK2]. The concept of a d-basis, originally considered in this context in [AVK], has been applied so far only to vector lattices with a large amount of projection bands. The absence of the projection bands has been the major obstacle for extending, otherwise very useful concept of d-bases, to arbitrary vector lattices. In Section 4 we will be able to overcome this obstacle by finding a new way to introduce d-independence in an arbitrary vector lattice. This allows us to produce a new definition of a d-basis which is free of the existence of projection bands. We illustrate this by proving several results devoted to cardinality of d-bases. Theorems 4.13 and 4.15 are the main of them and they assert that, under very general conditions, a vector lattice either has a singleton d-basis or else this d-basis must be infinite. This extends some of our work in [AK2, Section 6].

To make the reading of the article as much independent of [AK2] as possible we collect in the next section some necessary definitions and facts about d-bases. Most of this preliminary material, as well as some appropriate history regarding the subject, can be found in [AK2].

2. SOME PRELIMINARIES REGARDING D-BASES

In terminology regarding vector lattices we follow [AB]. All vector lattices in this work are assumed to be Archimedean. Whenever B is a projection band in a vector lattice we denote by P_B or $[B]$ the band-projection on B .

Recall [AK2, Definition 4.2] that a vector lattice X has a **cofinal family of projection bands** if for each non-zero band B in X there is a non-zero projection band $B_1 \subseteq B$. Under a different name the same concept was originally introduced in [LZ, Definition 30.3].

Definition 2.1. Let X be a vector lattice with a cofinal family of projection bands. A collection of vectors $\{e_\gamma : \gamma \in \Gamma\} \in X$ is said to be **d-independent** if for each projection band B in X the set $\{P_B e_\gamma : P_B e_\gamma \neq 0, \gamma \in \Gamma\}$ is linearly independent,

that is, the collection of all non-zero projections of the elements e_γ on B is linearly independent. Any maximal (by inclusion) set of d-independent vectors is called a **d-basis**.

A straightforward application of Zorn's lemma shows that in any vector lattice with a cofinal family of projection bands (in particular, in any Dedekind complete vector lattice and in any vector lattice with the projection property) there exists a d-basis.

We will explain now what type of representation the elements in X have when a d-basis $\{e_\gamma\}_{\gamma \in \Gamma}$ is fixed. Namely, for each $x \in X$ there is a full collection $\{X_i\}_{i \in I}$ of pairwise disjoint projection bands (depending on x) such that for each index i the set $\Gamma_i = \{\gamma \in \Gamma : [X_i]e_\gamma \neq 0\}$ is finite and the element $[X_i]x$ is a linear combination of these linearly independent projections $[X_i]e_\gamma$ with $\gamma \in \Gamma_i$, i.e., for some scalars $\lambda_\gamma^{(i)}$ we have

$$[X_i]x = \sum_{\gamma \in \Gamma_i} \lambda_\gamma^{(i)} [X_i]e_\gamma. \quad (1)$$

It is precisely the possibility of such a representation that justifies our use of the term “basis” here. We would like to stress that there is a drastic difference between the concepts of a Hamel basis and a d-basis. For instance, the cardinality of the latter can be much smaller. In extreme cases a d-basis may have only one element.

To be able to utilize d-bases more effectively we need to recall the operation of the complete union that is defined in [V, Chapter 4] as follows: *if (x_i) is a collection of pairwise disjoint elements in a vector lattice X such that there exist $\sup_i x_i^+$ and $\sup_i x_i^-$ in X , then the element $\sup_i x_i^+ - \sup_i x_i^-$ is called the **complete union** and is denoted by $\mathbf{S}_i x_i$. In particular, if each $x_i \in X_+$, then $\mathbf{S}_i x_i$ coincides with $\sup_i x_i$.*

When combined with operation \mathbf{S} , representation (1) described above gives the following. Fix an arbitrary $x \in X$. As said before we can find a full collection $\{X_i\}$ of pairwise disjoint projection bands in X and some scalars $\lambda_\gamma^{(i)}$ (all depending on x) such that for each i only a finite number of coefficients $\lambda_\gamma^{(i)}$ may be non-zero and for them $[X_i]x = \sum_\gamma \lambda_\gamma^{(i)} [X_i]e_\gamma$. Since the bands $\{X_i\}$ are pairwise disjoint and full in X we necessarily have $x = \mathbf{S}_i [X_i]x$, and hence the following “global” representation holds:

$$x = \mathbf{S}_i \sum_{\gamma \in \Gamma} \lambda_\gamma^{(i)} [X_i]e_\gamma. \quad (\star)$$

We will refer to representation (\star) as a **d-expansion** of x (with respect to the d-basis $\{e_\gamma\}$).

Formally speaking, a d-expansion is not unique since we can always subdivide any projection band X_i into the direct sum of two complementary projection bands (assuming, of course, that the band X_i is not one-dimensional). Essentially, however,

any d-expansion is unique in the following sense. If $x = \mathbf{S} \sum_j \lambda_\gamma^{(j)} [B_j] e_\gamma$ is another d-expansion with a generating collection of mutually disjoint bands $\{B_j\}_{j \in J}$, then necessarily $\lambda_\gamma^{(i)} = \lambda_\gamma^{(j)}$ whenever $[X_i \cap B_j] e_\gamma \neq 0$.

If we do not assume that X has a cofinal family of band-projections, then we cannot guarantee any longer the existence of a sufficient quantity of band-projections $[B] e_\gamma$ and, as a result of it, we loose the possibility of having a very useful d-expansion (\star) . We will show in Section 4 how to retain an analogue of the d-expansion for an arbitrary vector lattice X .

3. CHARACTERIZATION OF BAND PRESERVING PROJECTIONS

Representation (\star) was crucial in [AVK] for producing an example of a band preserving operator that was not a band-projection. Recently, in [AK2], modifying slightly the same idea, we have improved the previous example by constructing a band preserving **projection** operator that is not a band-projection. In view of the central role played by such projection operators in many situations it seems desirable to get a deeper understanding of their structure. And, as it turns out, the language of d-bases is adequate for obtaining a complete description of all band preserving projection operators on Dedekind complete vector lattices.

A vector sublattice X_0 of a vector lattice X is called **component-wise closed** in X if for each $u \in X_0$ the set $\mathcal{C}(u)$ of all components of u in X is a subset of X_0 . Recall that $\mathcal{C}(u) = \{v \in X : |v| \wedge |u - v| = 0\}$.

Observe that each component-wise closed vector sublattice of a vector lattice with the principal projection property (resp. the projection property) also satisfies the principal projection property (resp. the projection property). In particular, if X_0 is a component-wise closed vector sublattice of a Dedekind complete vector lattice, then X_0 has a cofinal family of band-projections.

It is proved in Proposition 4.9 in [AK2] that *if T is a disjointness preserving operator from a vector lattice X into a vector lattice Y , then for each ideal Y_0 in Y its inverse image $X_0 = T^{-1}(Y_0)$ is a vector sublattice of X and, moreover, X_0 is component-wise closed in X .*

A simple but useful Lemma 7.3 in [AK2] asserts that *if X is a vector lattice with a cofinal family of band-projections, then a linear operator $T : X \rightarrow X$ is band preserving if and only if it commutes with band-projections, that is, $TP = PT$ for each band-projection P on X .*

These two results will be used in our characterization of band preserving projection operators on laterally complete (in particular, universally complete) vector lattices. But first we present a useful characterization of band-projections.

Proposition 3.1. *Let X be a vector lattice and P be a band preserving projection operator on X . The operator P is a band-projection if and only if its kernel $P^{-1}(0)$ is a projection band.*

Proof. Only the “if” part is non-trivial. Assume that $A = P^{-1}(0)$ is a projection band in X . Consider the complimentary band $B = A^d$. Then $A \oplus B = X$. Since P is band preserving it follows that P leaves B invariant. But then the restriction of the operator P to B is a projection operator with the trivial kernel, and so P is the identity on B . ■

Recall that a vector lattice is **laterally complete** if each collection of pairwise disjoint elements has a supremum. By a well known theorem of Veksler and Geyler [VG, Theorem 8] each laterally complete vector lattice necessarily has the projection property. The most important example of laterally complete vector lattices is provided by the class of universally complete vector lattices.

Theorem 3.2. *Let W be a laterally complete vector lattice. There is a one-to-one correspondence between band preserving projection operators on W and vector sublattices of W which are **component-wise closed** and **laterally complete**. This correspondence is given by the map $P \mapsto P^{-1}(0)$, where P is an arbitrary band preserving projection operator on W .*

Proof. Let P be a band preserving projection operator on W . Since P preserves disjointness, Proposition 4.9 in [AK2] cited above implies that the kernel $W_0 = P^{-1}(0)$ is a component-wise closed vector sublattice of W . The assumptions that W is laterally complete and that P is band preserving imply immediately that W_0 is also laterally complete.

Conversely, let W_0 be a component-wise closed and laterally complete vector sublattice of W . Hence W_0 has the projection property. Therefore, as noted after Definition 2.1, there is a d-basis $\{u_\alpha\}$ in W_0 . Using that W_0 is component-wise closed we can easily see that $\{u_\alpha\}$ remains d-independent in W . Hence, by Zorn’s lemma, we can extend $\{u_\alpha\}$ to a d-basis in W , that is, to find elements $\{v_\beta\}$ in W such that $\{u_\alpha\} \cup \{v_\beta\}$ is a d-basis in W . Now we are ready to define a necessary operator P on X . Take an arbitrary $x \in W$ and consider its d-expansion (\star) with respect to the d-basis $\{u_\alpha\} \cup \{v_\beta\}$:

$$x = \mathbf{S}_i \left(\Sigma_\alpha \lambda_\alpha^{(i)} [X_i] u_\alpha + \Sigma_\beta \lambda_\beta^{(i)} [X_i] v_\beta \right).$$

The image Px is defined by “ignoring” the contribution of the first part of the d-basis, that is,

$$Px := \mathbf{S}_i \Sigma_\beta \lambda_\beta^{(i)} [X_i] v_\beta.$$

Since W is laterally complete and the elements $(\Sigma_\beta \lambda_\beta^{(i)} [X_i] v_\beta)_i$ are pairwise disjoint their complete union (that is, the value of P at x) exists. We omit a straightforward verification that P is a well defined band preserving projection operator on W and that $\ker(P) = W_0$. It is worth pointing out that the assumption that W_0 is laterally complete is essential for the validity of the equality $\ker(P) = W_0$.

Note that $I - P$ is also a band preserving projection operator and so, by the first part of the theorem, the kernel $\ker(I - P)$ or, equivalently, the range of P is a component-wise closed laterally complete vector sublattice of W . Hence, there are d-bases in $\ker(I - P)$.

It remains to prove that if P_1 and P_2 are two band preserving projection operators such that $\ker(P_1) = \ker(P_2)$, then $P_1 = P_2$. This is a very special feature of band preserving projections that does not hold for general projections. Let $\{u_\alpha\}$ be a d-basis in $\ker(P_1)$ and $\{v_\beta\}$ be a d-basis in $\ker(I - P_1)$. Let us verify that the system $\{u_\alpha\} \cup \{v_\beta\}$ is d-independent. Assume that for some band B in W we have

$$\sum_i \lambda_i [B]u_{\alpha_i} + \sum_j \mu_j [B]v_{\beta_j} = 0 \quad (2)$$

Let us apply P_1 to this identity, keeping in mind that P_1 commutes with $[B]$ and that $P_1(u_\alpha) = 0$ for each α and $P_1(v_\beta) = v_\beta$ for each β . We obtain that

$$\sum_j \mu_j [B]v_{\beta_j} = 0,$$

implying that $\mu_j = 0$ whenever $[B]v_{\beta_j} \neq 0$ since $\{v_\beta\}$ are d-independent. Similarly, applying to (2) operator $I - P_1$ we will conclude that $\lambda_i = 0$ whenever $[B]u_{\alpha_i} \neq 0$.

Now, the identity $P_1x + (I - P_1)x = x$ implies that $\{u_\alpha\} \cup \{v_\beta\}$ is, in fact, a d-basis in W .

Take any $z \in \ker(I - P_2)$. Since $\ker(P_1) = \ker(P_2)$ it follows (similarly to our arguments above) that the system $\{u_\alpha, z\}$ is d-independent. Therefore z allows a d-decomposition with respect to $\{v_\beta\}$ and hence $z \in \ker(I - P_1)$. Thus we see that $\ker(I - P_2) \subseteq \ker(I - P_1)$, and similarly $\ker(I - P_1) \subseteq \ker(I - P_2)$. This establishes that $P_1 = P_2$. ■

Recall [AK2] that a Dedekind complete vector lattice X is said to be **principally universally complete** if each principal band in X is universally complete.

Corollary 3.3. *Let X be a principally universally complete vector lattice. There is a one-to-one correspondence between band preserving projections on X and vector sublattices X_0 of X satisfying the following two conditions:*

- 1) X_0 is component-wise closed and
- 2) For each principal band B in X the intersection $B \cap X_0$ is laterally complete.

Now we are able to give a complete description of band preserving projections on arbitrary Dedekind complete lattices.

Theorem 3.4. *Let X be a Dedekind complete vector lattice and P be a band preserving projection operator on X . Then $X = X_1 + X_2$, where the uniquely determined complimentary bands X_1 and X_2 satisfy the following properties.*

- 1) X_1 is the maximal band such that the restriction of P to X_1 is a regular operator, and therefore $P|_{X_1}$ is a band-projection.

2) X_2 is principally universally complete, and hence the restriction $P|_{X_2}$ is described by the previous corollary.

Proof. Let X_1 be the maximal band of regularity of P . The existence of this band was established by de Pagter [P], and has been reproved in Theorem 14.8 in [AK2]. The latter theorem asserts also that the complimentary band $X_2 = (X_1)^d$ is principally universally complete. An application of Corollary 3.3 to the band X_2 finishes the proof. ■

Comparing Theorems 3.2 and 3.4 one is led to ask whether the latter theorem can be generalized to vector lattices with the projection property. Somewhat unexpectedly the answer to this question is negative. Let us explain why. Theorem 3.4 implies that *if a Dedekind complete vector lattice X does not have a non-trivial principally universally complete band, then each band preserving projection operator on X is a band-projection*. Therefore if we can produce an example of a band preserving projection operator P on a normed vector lattice X such that i) X has the projection property and ii) P is not a band-projection, then this will establish that a generalization in question cannot be true. (Keep in mind that no normed vector lattice can have a non-trivial principally universally complete band.) An example like that can be easily constructed directly or by modifying slightly the example given in [AK1, Theorem 2] and [AK2, Theorem 13.1]. A less direct proof that such a generalization is not possible is as follows. If it were true, then on each vector lattice without non-trivial principally universally complete bands each band preserving projection operator would be a band-projection. In other words, each vector lattice X like that would have a determining family of band-projections [AK2, Definition 7.2]. But then, by Theorem 8.5 in [AK2], each disjointness preserving bijection T from X onto an arbitrary vector lattice Y with a cofinal family of band-projections would have a disjointness preserving inverse. This contradicts [AK1, Theorem 2] and [AK2, Theorem 13.1].

Thus, we have shown that Theorem 3.4 cannot be generalized to vector lattices with the projection property. In other words, without the (r_u) -completeness of the vector lattice the projection property alone is not enough for the validity of Theorem 3.4.

4. D-INDEPENDENCE

All previous work [AVK, AAK, AK2] on d-bases, in particular the definitions of d-independence and of d-basis, as introduced in Definition 2.1, depends heavily on the availability of sufficiently many projection bands in the corresponding vector lattices. This means that these definitions cannot be easily adopted to arbitrary vector lattices. At the same time, the importance of the results devoted to d-bases and their numerous applications suggest that it would be desirable to extend these results to arbitrary vector lattices. To do so one has to avoid using projection bands

in the definition of d-independence. As we will see, this is possible and will be done in this section.

Definition 4.1. A system of elements $\{x_\gamma\}_{\gamma \in \Gamma}$ in a vector lattice X is called d-independent if for every band B in X , for every finite set of indices $\{\gamma_1, \dots, \gamma_n\} \subseteq \Gamma$, and every finite set of non-zero scalars $\{c_1, \dots, c_n\}$ the following implication holds:

$$\text{If } \sum_{j=1}^n c_j x_{\gamma_j} \perp B, \text{ then } x_{\gamma_j} \perp B \text{ for each } j = 1, 2, \dots, n.$$

Replacing each band B above by its disjoint complement $E = B^d$, we obtain equivalently that a system $\{x_\gamma\}_{\gamma \in \Gamma}$ is d-independent if and only if for every band E in X , for every finite set of indices $\{\gamma_1, \dots, \gamma_n\} \subseteq \Gamma$, and every finite set of non-zero scalars $\{c_1, \dots, c_n\}$ the following implication holds:

$$\text{If } \sum_{j=1}^n c_j x_{\gamma_j} \in E, \text{ then } x_{\gamma_j} \in E \text{ for each } j = 1, 2, \dots, n.$$

It is easy to see that **for any vector lattice X with a cofinal family of projection bands the above definition of d-independence is equivalent to that given in Definition 2.1.** Accordingly, we retain the term. To illustrate the difference between Definitions 2.1 and 4.1 consider the following example. Let $X = C[0, 1]$. Since there is no non-trivial projection band in $C[0, 1]$, any two linearly independent functions $x_1, x_2 \in X$ are d-independent in the old sense. However, this two functions can easily be d-dependent in the sense of our new definition. Indeed, if there is a point $t_0 \in [0, 1]$ such that $x_1(t) = x_2(t)$ for all t in a vicinity of t_0 , then clearly x_1 and x_2 are not d-independent in the sense of Definition 4.1.

More generally, a collection of functions $\{x_i\}_{i=1}^m$ in a vector lattice $X = C(K)$, where K is a compact Hausdorff space, fails to be d-independent if and only if there is a non-empty open subset U of K and scalars $\{c_i\}_{i=1}^m$, not all zero, such that $\sum_{i=1}^m c_i x_i(t) = 0$ for each $t \in U$. This leads us to the following result that will be used repeatedly.

Lemma 4.2. *For a function $x \in C(K)$, where K is a compact Hausdorff space, the following statements are equivalent:*

- 1) *For each $m \geq 2$ the collection x, x^2, \dots, x^m is d-independent.*
- 2) *For some $m \geq 2$ the collection x, x^2, \dots, x^m is d-independent.*
- 3) *For each non-empty open $U \subseteq K$ the restriction of x to U is not a constant.*

Proof. The implications $1) \Rightarrow 2) \Rightarrow 3)$ are obvious. To prove $3) \Rightarrow 1)$ assume, contrary to what we claim, that there is some $m \geq 2$ for which the collection x, x^2, \dots, x^m is not d-independent. Then there is a non-empty open subset V of K and some scalars c_i not all of which are zero such that $\sum_{i=1}^m c_i x^i(t) = 0$ for all t from V . Note, however, that $\sum_{i=1}^m c_i x^i(t) = 0$ is an algebraic equation of degree

not exceeding m . Therefore it cannot have more than m solutions, and this implies that the function x must be a constant on a non-empty open subset U of V , a contradiction. ■

Corollary 4.3. *For a function $x \in C(K)$ and a non-empty open subset U of K the following statements are equivalent:*

- 1) *For each $m \geq 2$ the collection of the restricted to U functions $x|_U, x^2|_U, \dots, x^m|_U$ is d -independent in the space $C(U)$.*
- 2) *For some $m \geq 2$ the collection of the restricted to U functions $x|_U, x^2|_U, \dots, x^m|_U$ is d -independent in the space $C(U)$.*
- 3) *For each non-empty open $V \subseteq U$ the restriction of x to V is not a constant.*

Corollary 4.4. *If a function $x \in C(K)$ satisfies Statement 3) of the previous corollary on a non-empty open subset U of K , then for each $m, k \geq 1$ and any scalars $\{\alpha_i\}_{i=1}^m$ and $\{\beta_i\}_{i=1}^k$ the functions $(\sum_{i=1}^m \alpha_i x^i)|_U$ and $(\sum_{i=1}^k \beta_i x^i)|_U$ are d -dependent in the space $C(U)$ if and only if $m = k$ and $\alpha_i = c\beta_i$ for some non-zero scalar c and for each i .*

The next statement is an immediate consequence of Zorn's lemma.

Proposition 4.5. *Let X be a vector lattice. There exists a maximal (by inclusion) d -independent system in X .*

As surprisingly as it may sound, in comparison with the case of vector lattices with a cofinal family of band-projections, it is unclear yet whether or not every maximal d -independent system in a vector lattice X is sufficient to obtain an analogue of d -expansion (\star) for elements in X . As we will show below in many cases it is sufficient. But in general we need to introduce a formal definition of a d -basis in an arbitrary vector lattice.

Definition 4.6. We say that a d -independent system $\{e_\gamma\}_{\gamma \in \Gamma}$ in a vector lattice X is a **d -basis** if for each $x \in X$ we can find a full system $\{X_i : i \in I\}$ of pairwise disjoint bands in X such that for each $i \in I$ there is a finite number of indices $\gamma_1, \dots, \gamma_n$ and non-zero scalars c_1, \dots, c_n (all depending on x and i) such that

$$x - \sum_{j=1}^n c_j e_{\gamma_j} \perp X_i. \quad (\star\star)$$

Obviously each d -basis is a maximal d -independent system but the validity of the converse statement is not known yet. If each band X_i in the previous definition is a projection band, then identity $(\star\star)$ can be rewritten as

$$[X_i]x = \sum_{j=1}^n c_i [X_i]e_{\gamma_j},$$

which is nothing else but d-expansion (\star). That is, we see that in the case **when X has a cofinal family of projection bands the new definition of a d-basis coincides with the old one.**

Problem 1. Is every maximal d-independent system a d-basis?

If the answer to this problem is negative, then the following two open problems are of considerable interest.

Problem 2. Describe the class of vector lattices admitting a d-basis.

Problem 3. Describe the class of vector lattices in which every maximal d-independent system is a d-basis.

Below we will introduce a condition under which the answer to Problem 1 is affirmative. The condition is rather technical but, nonetheless, it holds for many important classes of vector lattices.

Definition 4.7. Let x be a non-zero element in a vector lattice X . We say that a non-zero element $b \in X$ is a **semi-component** of x if there is a full in X system of pairwise disjoint bands $\{B_j : j \in J\}$ and a system of scalars $\{c_j : j \in J\}$ such that $b - c_j x \perp B_j$ for each $j \in J$.

Formally speaking, $b = 0$ can be also considered as a semi-component, but we are excluding this trivial case. Each non-zero component of x is obviously a semi-component as well. This shows that the notion of semi-components generalizes that of components.

As we will see below, it is of special importance when a given element has a semi-component in a given band. So, consider a band B in X and let $x \notin B^d$. If there is a non-zero component b of x that belongs to B , then b is a semi-component of x in B . In particular, the band-projection $b = [B]x$, whenever it exists, is a semi-component of x in B .

In general, it is easy to verify that a non-zero element $b \in B$ is a semi-component of x if there is a full in B system of pairwise disjoint bands $\{B_j : j \in J\}$ and a system of scalars $\{c_j : j \in J\}$ such that $b - c_j x \perp B_j$ for each $j \in J$.

Note also that the set of all semi-components of x in B , together with the zero vector, is a vector sublattice in X .

Example 4.8. *We describe here an important type of vector lattices that have plenty of semi-components in each band. At the same time, some of these vector lattices do not have a single non-trivial projection band.*

Consider any compact space K such that $X = C(K)$ has a dense subspace of essentially constant functions. (Following [AK2] we say that a function $f \in C(K)$ is **essentially constant** if for each non-empty open set $G \subseteq K$ there exists a non-empty open subset $G_1 \subseteq G$ such that f is constant on G_1 .) For instance,

each metrizable compact space K or, more generally, each compact space with the countable chain condition, satisfies this property (see [AB, Theorem 12.2], [HK, Theorem 0.1], [RR, Proposition, p. 130]).

Observe next that for an arbitrary non-zero band B in $C(K)$ there is a non-zero essentially constant function f in B . Indeed, fix an arbitrary function $g \in B$ such that $\|g\| = 1$ and $0 \leq g \leq 1$. By the hypothesis on K there is an essentially constant function $g' \in C(K)$ such that $\|g - g'\| < \varepsilon$ for some small $\varepsilon > 0$. Consider $f = (g' - \varepsilon \mathbf{1})^+$. We omit a trivial verification that f is essentially constant, non-zero and belongs to B .

To show that there are semi-components in any band B , take any $x_0 \notin B^d$ and take an arbitrary non-zero essentially constant function $f \in B$. The function fx_0 belongs to B and is a semi-component of x_0 .

If, additionally, the compact space K is connected, then X has no non-trivial projection band. ■

Definition 4.9. We will say that a vector lattice X satisfies condition $(*)$ if for every band B in X and every $x \notin B^d$ there exists a semi-component of x in B .

All vector lattices with the projection property, or with the principal projection property, or just with a cofinal family of band-projections satisfy $(*)$. Moreover, if a vector lattice X is such that *for each band B in X and for each element $x \notin B^d$ there exists a non-zero **component** of x that belongs to B* , then X also satisfies $(*)$. The class of vector lattices with this latter property contains properly the class of vector lattices with a cofinal family of band-projections. As we explained above, all vector lattices $C(K)$, where K is a metrizable compact space or a compact space with the countable chain condition, also satisfy $(*)$.

Proposition 4.10. *If a vector lattice X satisfies $(*)$ then each maximal d -independent system in X is a d -basis.*

Proof. Let $\{x_i\}$ be a maximal d -independent system in X . Assume, contrary to what we claim, that $\{x_i\}$ is not a d -basis. Then there exists an element $x \in X$ that cannot be d -expanded with respect to $\{x_i\}$ in the sense of Definition 4.6. That is, we cannot find a full in X collection of bands X_i satisfying $(\star\star)$. This implies that there exists a band B in X such that for any non-trivial band $B' \subset B$, for any finite set of indices $\{i_1, \dots, i_n\}$, and for any scalars $\lambda_1, \dots, \lambda_n$ the element $x - \sum_{k=1}^n \lambda_k x_{i_k}$ is not disjoint to B' . Let us consider this band B . Since X satisfies $(*)$, we can find a non-zero $b \in B$, a full in B system of pairwise disjoint bands $\{B_j\}$, and scalars c_j such that for each j we have $b - c_j x \perp B_j$. It is easy to verify now that the system $\{b, x_i\}$ is d -independent, a contradiction to the maximality of $\{x_i\}$. ■

An important class of vector lattices satisfying condition $(*)$ is described next. Recall that for each x in a vector lattice X there exists a compact Hausdorff space

K_x such that the principal ideal $X(x) = \{x' \in X : |x'| \leq \lambda|x|, \lambda \in \mathbb{R}\}$ is order isomorphic to an order dense vector sublattice of $C(K_x)$. We say that $X(x)$ is represented in $C(K_x)$. This representation is unique (up to a homeomorphism of K_x) if we require that the element x be mapped to the constant one function **1**.

Theorem 4.11. *Each (r_u) -complete vector lattice X with the countable sup property satisfies condition (*). In particular, every maximal d -independent system in X is a d -basis.*

Proof. To verify that X satisfies condition (*) take any band B in X and any element $x \in X_+$ that is not disjoint to B . Consider the principal ideal $X(x)$ generated by x . Then $X(x) = C(K)$ for some compact Hausdorff space K and $B \cap C(K)$ is a band in $C(K)$. Since X satisfies the countable sup property, the compact space K satisfies the countable chain condition and consequently, as said earlier, the collection of essentially constant functions is dense in $C(K)$ in view of [HK,RR]. As shown in Example 4.8 there is a non-zero essentially constant function f in $B \cap C(K)$. It remains to note that f is a semi-component of x not only in $C(K)$ but also in X , and we are done. ■

Now we are going to discuss some questions related to the cardinality of maximal d -independent systems and d -bases. More precisely we are interested in the following questions.

Problem 4. Let X be a vector lattice. Do all maximal d -independent systems in X have the same cardinality?

Problem 5. Let X be a vector lattice admitting a d -basis. Do all d -bases in X have the same cardinality?

Regardless of the answers to the previous problems, it will be also of interest to relate the cardinality of a maximal d -independent system (resp. of a d -basis) to some other cardinal characteristics of the vector lattice X , for example, to the disjointness type $t(X)$ introduced in [AV].

If a vector lattice X does not have a weak unit, then obviously any d -independent maximal system in X is infinite. The vector lattice c_{00} of eventually zero sequences provides an example of a discrete Dedekind complete vector lattice without a weak unit. Every d -basis in c_{00} is countable.

Discrete vector lattices with a weak unit provide the simplest example possible when there is a singleton d -basis. Discreteness, however, is not a decisive factor here. Each essentially one-dimensional vector lattice [AK2] with a weak unit has a singleton d -basis. Moreover, Gutman [G] constructed an example of an extremally disconnected compact Hausdorff space K without isolated points such that every continuous function on K is essentially constant. In other words, Gutman's space

$C(K)$ is atomless and, nevertheless, has a singleton d-basis. Another example of an atomless (but not Dedekind complete) vector lattice with a singleton d-basis is the space $C(\beta\mathbb{N} \setminus \mathbb{N})$. Similar examples can be found in [HK,RR]. It is an interesting open problem *to describe compact Hausdorff spaces K for which the vector lattice $C(K)$ has a singleton d-basis.*

If we do not impose any conditions on a vector lattice, then we can easily find a vector lattice X with a d-basis of a pre-assigned finite cardinality. Perhaps, the simplest example of a vector lattice with a d-basis of cardinality n , $1 \leq n < \infty$, is provided by the vector lattice X of piecewise polynomials of degree not exceeding n , that is, a continuous on $[0,1]$ function x belongs to X if and only if there is a partition $t_0 = 0 < t_1 < \dots < t_{m-1} < 1 = t_m$ of $[0,1]$ and polynomials p_1, \dots, p_m of degree no more than n such that $x \equiv p_j$ on $[t_{j-1}, t_j]$. This space was considered in [AK2].

If we do not restrict the degrees of the polynomials p_j above, then we obtain an example of a vector lattice with a countable d-basis. This vector lattice is a *subalgebra* of $C[0,1]$ and, as we shall demonstrate below, this additional algebraic structure is the actual reason of why any d-basis is infinite.

Two major results regarding cardinality were proved in [AK2]. Namely, in Theorem 6.8 we proved that a Dedekind complete vector lattice X either has a singleton d-basis or else any d-basis in X is infinite. After that we showed that for two large and important classes of vector lattices any d-basis is, in actuality, uncountable. One of this classes consists of all non-zero ideals in the space $L_0[0,1]$ (Theorem 6.10 in [AK2]) and the other class consists of all non-zero ideals in the universal completion of the $C(K)$ space, where K is any compact metric space without isolated points (Theorem 6.9 in [AK2]).

Below we will extend the first of these results to a much more general class of vector lattices than Dedekind complete. As for Theorems 6.9 and 6.10 in [AK2], it is presently unclear whether or not each infinite d-basis in an atomless Dedekind complete vector lattice must be uncountable.

Definition 4.12. Let us say that a vector lattice X is a **local algebra** if for each element $x \in X$ the principal ideal $X(x)$ can be represented as a **subalgebra** and a vector sublattice of the corresponding space $C(K_x)$.

There are many non-trivial examples of local algebras, for instance all (r_u) -complete vector lattices (in particular, all Dedekind complete vector lattices) are local algebras. Furthermore, a solution of the next problem can help find more such examples.

Problem 6. Is it possible to describe vector lattices that are local algebras in terms not involving representations?

Theorem 4.13. *Assume that a vector lattice X is a local algebra with no singleton d-basis. Then every d-basis in X , whenever it exists, is infinite.*

Proof. Suppose that e_1, \dots, e_n is a d-basis in X , where $n \geq 2$. There are at least two elements e_i, e_j , $1 \leq i < j \leq n$, which are not disjoint. Indeed, otherwise the element $e_1 + e_2 + \dots + e_n$ would be a singleton d-basis in X . Let $e = \sum_{k=1}^n |e_k|$ and let π be an order isomorphism of $X(e)$ onto a dense subalgebra of some $C(K)$ space such that $\pi e = \mathbf{1}$. Since πe_i and πe_j are d-independent, there exists a non-empty open subset $U \subset K$ such that at least one of the functions $\pi e_i, \pi e_j$ is not constant on each non-empty open subset $V \subset U$. Assume for definiteness that the function πe_i is such. Then for any positive integer m the functions $\pi e_i, \dots, (\pi e_i)^m$ belong to $\pi(X(e))$ and by Corollary 4.3 they are d-independent on U . To conclude the proof we will verify that this contradicts the fact that the system $\{\pi e_1, \dots, \pi e_n\}$ is a d-basis in $\pi(X(e))$. Indeed, the latter implies that for each non-empty open set $G \subseteq K$ and each $k = 1, \dots, m$ we can find a non-empty open subset of G on which the element $(\pi e_i)^k$ is a linear combination of the elements of our d-basis. Consequently we can find a non-empty open subset V of K on which each of the functions $\pi e_i, \dots, (\pi e_i)^m$ is a linear combination of the functions of the d-basis $\{\pi e_1, \dots, \pi e_n\}$. But this is impossible if $m > n$ since, as said above, the functions $\pi e_i, \dots, (\pi e_i)^m$ are linearly independent on V . ■

Corollary 4.14. *Let X be an (r_u) -complete vector lattice. Then either X has a singleton d-basis or every d-basis in X , whenever it exists, is infinite.*

We do not know if an analogue of the previous theorem remains true for d-maximal systems instead of d-bases. Under an additional condition we can prove this.

Theorem 4.15. *Assume that a vector lattice X is a local algebra with no singleton maximal d-independent system. If, additionally, X satisfies the countable sup property, i.e., every family of pairwise disjoint non-zero elements in X is at most countable, then every maximal d-independent system in X is infinite.*

Proof. Suppose to the contrary that there is a finite maximal d-independent system $\{x_1, \dots, x_n\}$ with $n \geq 2$. Then after representing $X(|x_1| + \dots + |x_n|)$ on a compact space K , exactly as in the proof of the previous theorem, we can find at least one element in our system $\{x_1, \dots, x_n\}$, let it be x_1 for definiteness, and a non-empty open subset U of K such that for any open $V \subset U$ and for any positive integer m the functions $(\pi x_1)^j$, $1 \leq j \leq m$, are linearly independent on V . Since X satisfies the countable sup property, the compact space K satisfies the countable chain condition.

Fix an $m > n + 1$. For each m -tuple of scalars $\bar{\alpha} = (\alpha_1, \dots, \alpha_m)$ consider the function

$$y_{\bar{\alpha}} = \sum_{i=1}^m \alpha_i (\pi x_1)^i.$$

Observe that if $\bar{\alpha}' = (\alpha'_1, \dots, \alpha'_m)$ is another m -tuple. then by Corollary 4.4 the functions $y_{\bar{\alpha}}$ and $y_{\bar{\alpha}'}$ are d -dependent on U iff $\bar{\alpha} \equiv c\bar{\alpha}'$ for some scalar $c \neq 0$.

Let us fix an arbitrary uncountable collection of points $\Lambda = \{\bar{\alpha}\} \subset \mathbb{R}^m$ such that any $n+1$ pairwise distinct points $\bar{\alpha}_1, \dots, \bar{\alpha}_{n+1}$ in Λ are linearly independent. Then, by Corollary 4.4, the functions $y_{\bar{\alpha}_1}, \dots, y_{\bar{\alpha}_{n+1}}$ are d -independent on U .

Fix any $\bar{\alpha} \in \Lambda$ and consider the function $y_{\bar{\alpha}}$ introduced above. Since $\{x_1, \dots, x_n\}$ is a maximal d -independent system, the function $y_{\bar{\alpha}}$ cannot be d -independent of this system on U . Therefore, there is a non-empty open subset $U_{\bar{\alpha}} \subset U$ such that $y_{\bar{\alpha}}$ coincides on $U_{\bar{\alpha}}$ with a linear combination of the functions $\pi x_1, \dots, \pi x_n$.

We claim next that whenever we take arbitrary $n+1$ mutually distinct points $\bar{\alpha}_1, \dots, \bar{\alpha}_{n+1} \in \Lambda$ we necessarily have that

$$\bigcap_{j=1}^{n+1} U_{\bar{\alpha}_j} = \emptyset \quad (3).$$

Indeed, if an open set $V = \bigcap_j U_{\bar{\alpha}_j}$ is not empty, then on V each of the functions $y_{\bar{\alpha}_j}$ is a linear combination of the functions from $\pi x_1, \dots, \pi x_n$. On the other hand, as said above, these functions $y_{\bar{\alpha}_j}$ are linearly independent on V , a contradiction.

We conclude the proof by showing that the existence of an uncountable family $\{U_{\bar{\alpha}} : \bar{\alpha} \in \Lambda\}$ of non-empty open sets satisfying (3), contradicts the countable chain condition in K . We will use induction on n .

Assume that the above statement is true for some n , and let us prove it for $n+1$. Consider all sets of the form

$$V_{\beta} := U_{\bar{\alpha}_1} \cap \dots \cap U_{\bar{\alpha}_{n+1}}.$$

where $\beta = (\bar{\alpha}_1, \dots, \bar{\alpha}_{n+1})$ and $\alpha_1, \dots, \alpha_{n+1}$ are pairwise distinct points in Λ . Observe that by (3) the sets V_{β} are pairwise disjoint. If each $V_{\beta} = \emptyset$, then we are done in view of the induction hypothesis. So some sets V_{β} are non-empty, and, since they are pairwise disjoint, there may be at most countably many of such sets. Let $V_{\beta_1}, \dots, V_{\beta_k}, \dots$ be all these non-empty sets. Since each $\beta_k = (\bar{\alpha}_1^{(k)}, \dots, \bar{\alpha}_{n+1}^{(k)})$ depends on a finite number of indices in Λ , there are uncountably many α -s that have not been used. Consider all the sets V_{β} for which β depends on at least one of these unused α -s. Again by the induction hypothesis, all these V_{β} cannot be empty. Take an arbitrary such V_{β} that is not empty. However, this V_{β} must be disjoint from each V_{β_k} , and so we have a contradiction. ■

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